16

Relations and Functions

TOPIC 1 Types of Relations, Inverse of a Relation, Mappings, Mapping of Functions, Kinds of Mapping of Functions

1. Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Then the number of elements in the set $C = \{f : A \to B \mid 2 \in f(A)$ and f is not one-one$\}$ is ___________. **[NA Sep. 05, 2020 (II)]**

2. Let a function $f:(0,\infty)\to(0,\infty)$ be defined by $f(x)=\left|1-\frac{1}{x}\right|$. Then f is : **[Jan. 11, 2019 (II)]**

(a) not injective but it is surjective
(b) injective only
(c) neither injective nor surjective
(d) both injective as well as surjective

3. The number of functions f from $\{1, 2, 3, ..., 20\}$ onto $\{1, 2, 3, ..., 20\}$ such that $f(k)$ is a multiple of 3, whenever k is a multiple of 4 is : **[Jan. 11, 2019 (II)]**

(a) $6^5 \times (15)!$ (b) $5! \times 6!$
(c) $(15)! \times 6!$ (d) $5^6 \times 15$

4. Let **N** be the set of natural numbers and two functions f and g be defined as $f, g : \mathbf{N} \to \mathbf{N}$ such that

$$f(n)=\begin{cases}\frac{n+1}{2} & \text{if n is odd}\\ \frac{n}{2} & \text{if n is even}\end{cases}$$

and $g(n) = n - (-1)^n$. Then fog is: **[Jan. 10, 2019 (II)]**

(a) onto but not one-one.
(b) one-one but not onto.
(c) both one-one and onto.
(d) neither one-one nor onto.

5. Let $A = \{x \in \mathbf{R} : x$ is not a positive integer$\}$. Define a function f: $A \to \mathbf{R}$ as $f(x) = \frac{2x}{x-1}$, then f is: **[Jan. 09, 2019 (II)]**

(a) not injective
(b) neither injective nor surjective
(c) surjective but not injective
(d) injective but not surjective

6. The function $f : R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as $f(x) = \frac{x}{1+x^2}$, is : **[2017]**

(a) neither injective nor surjective
(b) invertible
(c) injective but not surjective
(d) surjective but not injective

7. The function f: $N \to N$ defined by $f(x) = x - 5\left[\frac{x}{5}\right]$, where N is set of natural numbers and $[x]$ denotes the greatest integer less than or equal to x, is : **[Online April 9, 2017]**

(a) one-one and onto.
(b) one-one but not onto.
(c) onto but not one-one.
(d) neither one-one nor onto.

8. Let $A = \{x_1, x_2, ..., x_7\}$ and $B = \{y_1, y_2, y_3\}$ be two sets containing seven and three distinct elements respectively. Then the total number of functions $f : A \to B$ that are onto, if there exist exactly three elements x in A such that $f(x) = y_2$, is equal to : **(Online April 11, 2015)**

(a) $14.{}^7C_3$ (b) $16.{}^7C_3$ (c) $14.{}^7C_2$ (d) $12.{}^7C_2$

9. Let $f : R \to R$ be defined by $f(x) = \frac{|x|-1}{|x|+1}$ then f is: **[Online April 19, 2014]**

(a) both one-one and onto
(b) one-one but not onto
(c) onto but not one-one
(d) neither one-one nor onto.

10. Let P be the relation defined on the set of all real numbers such that
$P = \{(a, b) : \sec^2 a - \tan^2 b = 1\}$. Then P is: **[Online April 9, 2014]**
(a) reflexive and symmetric but not transitive.
(b) reflexive and transitive but not symmetric.
(c) symmetric and transitive but not reflexive.
(d) an equivalence relation.

11. Let $R = \{(x, y) : x, y \in N \text{ and } x^2 - 4xy + 3y^2 = 0\}$, where N is the set of all natural numbers. Then the relation R is : **[Online April 23, 2013]**
(a) reflexive but neither symmetric nor transitive.
(b) symmetric and transitive.
(c) reflexive and symmetric,
(d) reflexive and transitive.

12. Let $R = \{(3,3)(5,5),(9,9),(12,12),(5,12),(3,9),(3,12),(3,5)\}$ be a relation on the set $A = \{3, 5, 9, 12\}$. Then, R is : **[Online April 22, 2013]**
(a) reflexive, symmetric but not transitive.
(b) symmetric, transitive but not reflexive.
(c) an equivalence relation.
(d) reflexive, transitive but not symmetric.

13. Let $A = \{1, 2, 3, 4\}$ and $R : A \to A$ be the relation defined by $R = \{(1, 1), (2, 3), (3, 4), (4, 2)\}$. The correct statement is : **[Online April 9, 2013]**
(a) R does not have an inverse.
(b) R is not a one to one function.
(c) R is an onto function.
(d) R is not a function.

14. If $P(S)$ denotes the set of all subsets of a given set S, then the number of one-to-one functions from the set $S = \{1, 2, 3\}$ to the set $P(S)$ is **[Online May 19, 2012]**
(a) 24 (b) 8 (c) 336 (d) 320

15. If $A = \{x \in z^+ : x < 10$ and x is a multiple of 3 or 4$\}$, where z^+ is the set of positive integers, then the total number of symmetric relations on A is **[Online May 12, 2012]**
(a) 2^5 (b) 2^{15} (c) 2^{10} (d) 2^{20}

16. Let R be the set of real numbers. **[2011]**
Statement-1: $A = \{(x, y) \in R \times R : y - x \text{ is an integer}\}$ is an equivalence relation on R.
Statement-2: $B = \{(x, y) \in R \times R : x = \alpha y$ for some rational number $\alpha\}$ is an equivalence relation on R.
(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is false.
(c) Statement-1 is false, Statement-2 is true.
(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

17. Consider the following relations:
R = $\{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$; $S = \{\left(\frac{m}{n}, \frac{p}{q}\right) \mid m, n, p$ and q are integers such that $n, q \neq 0$ and $qm = pn\}$.
Then **[2010]**
(a) Neither R nor S is an equivalence relation
(b) S is an equivalence relation but R is not an equivalence relation
(c) R and S both are equivalence relations
(d) R is an equivalence relation but S is not an equivalence relation

18. Let R be the real line. Consider the following subsets of the plane $R \times R$:
$S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$
$T = \{(x, y) : x - y \text{ is an integer}\}$,
Which one of the following is true? **[2008]**
(a) Neither S nor T is an equivalence relation on R
(b) Both S and T are equivalence relation on R
(c) S is an equivalence relation on R but T is not
(d) T is an equivalence relation on R but S is not

19. Let W denote the words in the English dictionary. Define the relation R by $R = \{(x, y) \in W \times W |$ the words x and y have at least one letter in common.$\}$ Then R is **[2006]**
(a) not reflexive, symmetric and transitive
(b) relexive, symmetric and not transitive
(c) reflexive, symmetric and transitive
(d) reflexive, not symmetric and transitive

20. Let $R = \{(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)\}$ be a relation on the set $A = \{3, 6, 9, 12\}$. The relation is **[2005]**
(a) reflexive and transitive only
(b) reflexive only
(c) an equivalence relation
(d) reflexive and symmetric only

21. Let $f : (-1, 1) \to B$, be a function defined by $f(x) = \tan^{-1}\frac{2x}{1 - x^2}$, then f is both one - one and onto when B is the interval **[2005]**
(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left[0, \frac{\pi}{2}\right)$
(c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

22. Let $R = \{(1,3),(4,2),(2,4),(2,3),(3,1)\}$ be a relation on the set $A = \{1, 2, 3, 4\}$.. The relation R is **[2004]**
(a) reflexive (b) transitive
(c) not symmetric (d) a function

23. If $f : R \to S$, defined by $f(x) = \sin x - \sqrt{3}\cos x + 1$, is onto, then the interval of S is **[2004]**

(a) $[-1, 3]$ (b) $[-1, 1]$ (c) $[0, 1]$ (d) $[0, 3]$

24. A function f from the set of natural numbers to integers defined by **[2003]**

$$f(n) = \begin{cases} \dfrac{n-1}{2}, \text{when n is odd} \\ -\dfrac{n}{2}, \text{when n is even} \end{cases} \quad \text{is}$$

(a) neither one -one nor onto

(b) one-one but not onto

(c) onto but not one-one

(d) one-one and onto both.

TOPIC 2 Composite Functions & Relations, Inverse of a Function, Binary Operations

25. The inverse function of $f(x) = \dfrac{8^{2x} - 8^{-2x}}{8^{2x} + 8^{-2x}}, x \in (-1, 1)$, is ________. **[Jan. 8, 2020 (I)]**

(a) $\dfrac{1}{4}\log_e\left(\dfrac{1+x}{1-x}\right)$ (b) $\dfrac{1}{4}(\log_8 e)\log_e\left(\dfrac{1-x}{1+x}\right)$

(c) $\dfrac{1}{4}\log_e\left(\dfrac{1-x}{1+x}\right)$ (d) $\dfrac{1}{4}(\log_8 e)\log_e\left(\dfrac{1+x}{1-x}\right)$

26. If $g(x) = x^2 + x - 1$ and $(gof)(x) = 4x^2 - 10x + 5$, then $f\left(\dfrac{5}{4}\right)$ is equal to: **[Jan. 7, 2020 (I)]**

(a) $\dfrac{3}{2}$ (b) $-\dfrac{1}{2}$ (c) $\dfrac{1}{2}$ (d) $-\dfrac{3}{2}$

27. For a suitably chosen real constant a, let a function, $f : R - \{-a\} \to R$ be defined by $f(x) = \dfrac{a - x}{a + x}$. Further suppose that for any real number $x \neq -a$ and $f(x) \neq -a$, $(fof)(x) = x$. Then $f\left(-\dfrac{1}{2}\right)$ is equal to: **[Sep. 06, 2020 (II)]**

(a) $\dfrac{1}{3}$ (b) $-\dfrac{1}{3}$ (c) -3 (d) 3

28. For $x \in \left(0, \dfrac{3}{2}\right)$, let $f(x) = \sqrt{x}$, $g(x) = \tan x$ and $h(x) = \dfrac{1-x^2}{1+x^2}$. If $\phi(x) = ((hof)og)(x)$, then $\phi\left(\dfrac{\pi}{3}\right)$ is equal to : **[April 12, 2019 (I)]**

(a) $\tan\dfrac{\pi}{12}$ (b) $\tan\dfrac{11\pi}{12}$ (c) $\tan\dfrac{7\pi}{12}$ (d) $\tan\dfrac{5\pi}{12}$

29. Let $f(x) = x^2, x \in R$. For any $A \subseteq R$, define $g(A) = \{x \in R : f(x) \in A\}$. If $S = [0, 4]$, then which one of the following statements is not true ? **[April 10, 2019 (I)]**

(a) $g(f(S)) \neq S$ (b) $f(g(S)) = S$

(c) $g(f(S)) = g(S)$ (d) $f(g(S)) \neq f(S)$

30. For $x \in \mathbf{R} - \{0, 1\}$, let $f_1(x) = \dfrac{1}{x}$, $f_2(x) = 1 - x$ and $f_3(x) = \dfrac{1}{1-x}$ be three given functions. If a function, $J(x)$ satisfies $(f_2 o J o f_1)(x) = f_3(x)$ then $J(x)$ is equal to: **[Jan. 09, 2019 (I)]**

(a) $f_3(x)$ (b) $\dfrac{1}{x}f_3(x)$ (c) $f_2(x)$ (d) $f_1(x)$

31. Let N denote the set of all natural numbers. Define two binary relations on N as $R_1 = \{(x, y) \in N \times N : 2x + y = 10\}$ and $R_2 = \{(x, y) \in N \times N : x + 2y = 10\}$. Then **[Online April 16, 2018]**

(a) Both R_1 and R_2 are transitive relations

(b) Both R_1 and R_2 are symmetric relations

(c) Range of R_2 is $\{1, 2, 3, 4\}$

(d) Range of R_1 is $\{2, 4, 8\}$

32. Consider the following two binary relations on the set $A = \{a, b, c\} : R_1 = \{(c, a)(b, b), (a, c), (c, c), (b, c), (a, a)\}$ and $R_2 = \{(a, b), (b, a), (c, c), (c, a), (a, a), (b, b), (a, c)$. Then **[Online April 15, 2018]**

(a) R_2 is symmetric but it is not transitive

(b) Both R_1 and R_2 are transitive

(c) Both R_1 and R_2 are not symmetric

(d) R_1 is not symmetric but it is transitive

33. Let $f : A \to B$ be a function defined as $f(x) = \dfrac{x-1}{x-2}$, where $A = R - \{2\}$ and $B = R - \{1\}$. Then f is **[Online April 15, 2018]**

(a) invertible and $f^{-1}(y) = \dfrac{2y+1}{y-1}$

(b) invertible and $f^{-1}(y) = \dfrac{3y-1}{y-1}$

(c) no invertible

(d) invertible and $f^{-1}(y) = \dfrac{2y-1}{y-1}$

34. Let $f(x)=2^{10}\cdot x+1$ and $g(x)=3^{10}\cdot x-1$. If (fog)(x)=x, then x is equal to : **[Online April 8, 2017]**

(a) $\frac{3^{10}-1}{3^{10}-2^{-10}}$ (b) $\frac{2^{10}-1}{2^{10}-3^{-10}}$

(c) $\frac{1-3^{-10}}{2^{10}-3^{-10}}$ (d) $\frac{1-2^{-10}}{3^{10}-2^{-10}}$

35. For $x \in R, x \neq 0$, let $f_0(x)=\frac{1}{1-x}$ and $f_{n+1}(x)=f_0(f_n(x))$, $n=0, 1, 2, ...$ Then the value of $f_{100}(3)+f_1\left(\frac{2}{3}\right)+f_2\left(\frac{3}{2}\right)$ is equal to : **[Online April 9, 2016]**

(a) $\frac{8}{3}$ (b) $\frac{4}{3}$ (c) $\frac{5}{3}$ (d) $\frac{1}{3}$

36. If g is the inverse of a function f and $f'(x)=\frac{1}{1+x^5}$, then $g'(x)$ is equal to: **[2014]**

(a) $\frac{1}{1+\{g(x)\}^5}$ (b) $1+\{g(x)\}^5$

(c) $1+x^5$ (d) $5x^4$

37. Let A and B be non empty sets in R and $f: A \to B$ is a bijective function. **[Online May 26, 2012]**

Statement 1: f is an onto function.

Statement 2: There exists a function $g: B \to A$ such that $fog = I_B$.

(a) Statement 1 is true, Statement 2 is false.

(b) Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1.

(c) Statement 1 is false, Statement 2 is true.

(d) Statement 1 is true, Statement 2 is true, Statement 2 is not the correct explanation for Statement 1.

38. Let f be a function defined by

$f(x)=(x-1)^2+1, (x\geq 1)$. **[2011RS]**

Statement - 1 : The set $\{x : f(x)=f^{-1}(x)\}=\{1,2\}$.

Statement - 2 : f is a bijection and

$f^{-1}(x)=1+\sqrt{x-1}, x\geq 1$.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

(b) Statement-1 is true, Statement-2 is true; Statement-2 is NOT a correct explanation for Statement-1.

(c) Statement-1 is true, Statement-2 is false.

(d) Statement-1 is false, Statement-2 is true.

39. Let $f(x)=(x+1)^2-1, x\geq -1$

Statement -1 : The set $\{x : f(x)=f^{-1}(x)\}=\{0,-1\}$

Statement-2 : f is a bijection. **[2009]**

(a) Statement-1 is true, Statement-2 is true. Statement-2 is not a correct explanation for Statement-1.

(b) Statement-1 is true, Statement-2 is false.

(c) Statement-1 is false, Statement-2 is true.

(d) Statement-1 is true, Statement-2 is true. Statement-2 is a correct explanation for Statement-1.

40. Let $f: N\to Y$ be a function defined as $f(x)=4x+3$ where $Y=\{y \in N : y=4x+3 \text{ for some } x \in N\}$.

Show that f is invertible and its inverse is **[2008]**

(a) $g(y)=\frac{3y+4}{3}$ (b) $g(y)=4+\frac{y+3}{4}$

(c) $g(y)=\frac{y+3}{4}$ (d) $g(y)=\frac{y-3}{4}$

Hints & Solutions

1. **(19.00)**
The desired functions will contain either one element or two elements in its codomain of which '2' always belongs to $f(A)$.
$\therefore$ The set B can be $\{2\}, \{1, 2\}, \{2, 3\}, \{2, 4\}$
Total number of functions $= 1 + (2^3 - 2)3 = 19$.

2. **(Bonus)** $f : (0, \infty) \to (0, \infty)$

$f(x) = \left|1 - \frac{1}{x}\right|$ is not a function

$\because$ $f(1) = 0$ and $1 \in$ domain but $0 \notin$ co-domain

Hence, $f(x)$ is not a function.

3. **(c)** Domain and codomain $= \{1, 2, 3, \ldots, 20\}$.

There are five multiple of 4 as 4, 8, 12, 16 and 20.

and there are 6 multiple of 3 as 3, 6, 9, 12, 15, 18.

Since, when ever k is multiple of 4 then $f(k)$ is multiple of 3 then total number of arrangement

$= {}^6c_5 \times 5! = 6!$

Remaining 15 elements can be arranged in 15! ways.

Since, for every input, there is an output

$\Rightarrow$ function $f(k)$ in onto

$\therefore$ Total number of arrangement $= 15!\,.\,6!$

4. **(a)** $f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$

$$g(n) = \begin{cases} 2, & n = 1 \\ 1, & n = 2 \\ 4, & n = 3 \\ 3, & n = 4 \\ 6, & n = 5 \\ 5, & n = 6 \end{cases}$$

Then,

$$f(g(n)) = \begin{cases} \frac{g(n)+1}{2}, & \text{if } g(n) \text{ is odd} \\ \frac{g(n)}{2}, & \text{if } g(n) \text{ is even} \end{cases}$$

$$f(g(n)) = \begin{cases} 1, & n = 1 \\ 1, & n = 2 \\ 2, & n = 3 \\ 2, & n = 4 \\ 3, & n = 5 \\ 3, & n = 6 \\ \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \end{cases}$$

$\Rightarrow fog$ is onto but not one - one

5. **(d)** As $A = \{x \in R : x$ is not a positive integer$\}$

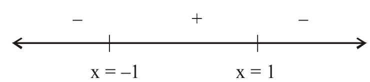
A function $f : A \to R$ given by $f(x) = \frac{2x}{x-1}$

$f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$
So, f is one-one.
As $f(x) \neq 2$ for any $x \in A \Rightarrow f$ is not onto.
Hence f is injective but not surjective.

6. **(d)** We have $f : R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$f(x) = \frac{x}{1+x^2}\ \forall x \in R$

$\Rightarrow f'(x) = \frac{(1+x^2).1 - x.2x}{(1+x^2)^2} = \frac{-(x+1)(x-1)}{(1+x^2)^2}$



sign of $f'(x)$
$\Rightarrow f'(x)$ changes sign in different intervals.
$\therefore$ Not injective

Now $y = \frac{x}{1+x^2}$
$\Rightarrow$ $y + yx^2 = x$
$\Rightarrow$ $yx^2 - x + y = 0$
For $y \neq 0$, $D = 1 - 4y^2 \geq 0$

$\Rightarrow$ $y \in \left[\frac{-1}{2}, \frac{1}{2}\right] - \{0\}$

For $y = 0 \Rightarrow x = 0$

$\therefore$ Range is $\left[\frac{-1}{2}, \frac{1}{2}\right]$

$\Rightarrow$ Surjective but not injective

7. **(d)** $\left.\begin{array}{l} f(1) = 1 - 5[1/5] = 1 \\ f(6) = 6 - 5[6/5] = 1 \end{array}\right\} \rightarrow$ Many one

$f(10) = 10 - 5(2) = 0$ which is not in co–domain.

Neither one–one nor onto.

8. **(a)** Number of onto function such that exactly three elements in $x \in A$ such that $f(x) = \frac{1}{2}$ is equal to

$= {}^7C_3, \{2^4 - 2\} = 14.\ {}^7C_3$

9. **(c)** $f(x) = \frac{|x|-1}{|x|+1}$

for one-one function if $f(x_1) = f(x_2)$ then x_1 must be equal to x_2

Let $f(x_1) = f(x_2)$

$$\frac{|x_1|-1}{|x_1|+1} = \frac{|x_2|-1}{|x_2|+1}$$

$$|x_1||x_2| + |x_1| - |x_2| - 1 = |x_1||x_2| - |x_1| + |x_2| - 1$$

$\Rightarrow |x_1| - |x_2| = |x_2| - |x_1|$

$2|x_1| = 2|x_2|$

$|x_1| = |x_2|$

$x_1 = x_2, x_1 = -x_2$

here x_1 has two values therefore function is many one function.

For onto : $f(x) = \frac{|x|-1}{|x|+1}$

for every value of $f(x)$ there is a value of x in domain set.

If $f(x)$ is negative then $x = 0$

for all positive value of $f(x)$, domain contain atleast one element. Hence $f(x)$ is onto function.

10. **(d)** $P = \{(a,b) : \sec^2 a - \tan^2 b = 1\}$

For reflexive :

$\sec^2 a - \tan^2 a = 1$ (true $\forall\ a$)

For symmetric :

$\sec^2 b - \tan^2 a = 1$

L.H.S

$1 + \tan^2 b - (\sec^2 a - 1) = 1 + \tan^2 b - \sec^2 a + 1$

$= -(\sec^2 a - \tan^2 b) + 2$

$= -1 + 2 = 1$

So, Relation is symmetric

For transitive :

if $\sec^2 a - \tan^2 b = 1$ and $\sec^2 b - \tan^2 c = 1$

$\sec^2 a - \tan^2 c = (1 + \tan^2 b) - (\sec^2 b - 1)$

$= -\sec^2 b + \tan^2 b + 2$

$= -1 + 2 = 1$

So, Relation is transitive.

Hence, Relation P is an equivalence relation

11. **(d)** $R = \{(x, y) : x, y \in N \text{ and } x^2 - 4xy + 3y^2 = 0\}$

Now, $x^2 - 4xy + 3y^2 = 0$

$\Rightarrow (x - y)(x - 3y) = 0$

$\therefore x = y$ or $x = 3y$

$\therefore R = \{(1, 1), (3, 1), (2, 2), (6, 2), (3, 3), (9, 3), ...\}$

Since (1, 1), (2, 2), (3, 3),...... are present in the relation, therefore R is reflexive.

Since (3, 1) is an element of R but (1, 3) is not the element of R, therefore R is not symmetric

Here $(3, 1) \in R$ and $(1, 1) \in R \Rightarrow (3, 1) \in R$

$(6, 2) \in R$ and $(2, 2) \in R \Rightarrow (6, 2) \in R$

For all such $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

Hence R is transitive.

12. **(d)** Let R = {(3, 3), (5, 5), (9, 9), (12, 12), (5, 12), (3, 9), (3, 12), (3, 5)} be a relation on set

A = {3, 5, 9, 12}

Clearly, every element of A is related to itself.

Therefore, it is a reflexive.

Now, R is not symmetry because 3 is related to 5 but 5 is not related to 3.

Also R is transitive relation because it satisfies the property that if a R b and b R c then a R c.

13. **(c)** Domain = {1, 2, 3, 4}

Range = {1, 2, 3, 4}

$\therefore$ Domain = Range

Hence the relation R is onto function.

14. **(c)** Let $S = \{1, 2, 3\} \Rightarrow n(S) = 3$

Now, $P(S)$ = set of all subsets of S

total no. of subsets $= 2^3 = 8$

$\therefore n[P(S)] = 8$

Now, number of one-to-one functions from $S \to P(S)$ is

$${}^8P_3 = \frac{8!}{5!} = 8 \times 7 \times 6 = 336.$$

15. **(b)** A relation on a set A is said to be symmetric iff $(a,b) \in A \Rightarrow (b,a) \in A,\ \forall\ a,b \in A$

Here $A = \{3, 4, 6, 8, 9\}$

Number of order pairs of $A \times A = 5 \times 5 = 25$

Divide 25 order pairs of $A \times A$ in 3 parts as follows :

Part – A : (3, 3), (4, 4), (6, 6), (8, 8), (9, 9)

Part – B : (3, 4), (3, 6), (3, 8), (3, 9), (4, 6), (4, 8),(4, 9), (6, 8), (6, 9), (8, 9)

Part – C : (4, 3), (6, 3), (8, 3),(9, 3), (6, 4), (8, 4), (9, 4), (8, 6), (9, 6), (9, 8)

In part – A, both components of each order pair are same.

In part – B, both components are different but not two such order pairs are present in which first component of one order pair is the second component of another order pair and vice-versa.

In part–C, only reverse of the order pairs of part –B are present i.e., if (a, b) is present in part – B, then (b, a) will be present in part –C

For example (3, 4) is present in part – B and (4, 3) present in part –C.

Number of order pair in A, B and C are 5, 10 and 10 respectively.

In any symmetric relation on set A, if any order pair of part –B is present then its reverse order pair of part –C will must be also present.

Hence number of symmetric relation on set A is equal to the number of all relations on a set D, which contains all the order pairs of part –A and part– B.

Now $n(D) = n(A) + n(B) = 5 + 10 = 15$

Hence number of all relations on set $D = (2)^{15}$

$\Rightarrow$ Number of symmetric relations on set $D = (2)^{15}$

16. **(a)** $\because x - x = 0 \in I$ ($\therefore R$ is reflexive)

Let $(x, y) \in R$ as $x - y$ and $y - x \in I$ ($\because R$ is symmetric)

Now $x - y \in I$ and $y - z \in I \Rightarrow x - z \in I$

So, R is transative.

Hence R is equivalence.

Similarly as $x = \alpha y$ for $\alpha = 1$. B is reflexive symmetric and transative. Hence B is equivalence.

Both relations are equivalence but not the correct explanation.

17. **(b)** Let $x R y$.

$$\Rightarrow x = wy \Rightarrow y = \frac{x}{w}$$

$\Rightarrow (y, x) \notin R$

R is not symmetric

Let $S : \frac{m}{n} S \frac{p}{q}$

$$\Rightarrow qm = pn \Rightarrow \frac{p}{q} = \frac{m}{n}$$

$$\because \frac{m}{n} = \frac{m}{n} \therefore \text{ reflexive.}$$

$$\frac{m}{n} = \frac{p}{q} \Rightarrow \frac{p}{q} = \frac{m}{n} \therefore \text{ symmetric}$$

Let $\frac{m}{n} S \frac{p}{q}, \frac{p}{q} S \frac{r}{s}$

$\Rightarrow qm = pn, ps = rq$

$$\Rightarrow \frac{p}{q} = \frac{m}{n} = \frac{r}{s}$$

$\Rightarrow ms = rn$ transitive.

S is an equivalence relation.

18. **(d)** Given that

$S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$

$\because x \neq x + 1$ for any $x \in (0, 2)$

$\Rightarrow (x, x) \notin$ S

So, S is not reflexive.

Hence, S in not an equivalence relation.

Given $T = \{x, y) : x - y$ is an integer$\}$

$\because x - x = 0$ is an integer, $\forall x \in R$

So, T is reflexive.

Let $(x, y) \in$ T $\Rightarrow$ $x - y$ is an integer then $y - x$ is also an integer $\Rightarrow (y, x) \in R$

$\therefore T$ is symmetric

If $x - y$ is an integer and $y - z$ is an integer then $(x - y) + (y - z) = x - z$ is also an integer.

$\therefore$ T is transitive

Hence T is an equivalence relation.

19. **(b)** Clearly $(x, x) \in R, \forall x \in W$

$\because$ All letter are common in some word. So R is reflexive.

Let $(x, y) \in R$, then $(y, x) \in R$ as x and y have at least one letter in common. So, R is symmetric.

But R is not transitive for example

Let x = BOY, y = TOY and z = THREE

then $(x, y) \in R$ (O, Y are common) and $(y, z) \in R$ (T is common) but $(x, z) \notin R$. (as no letter is common)

20. **(a)** R is reflexive and transitive only.

Here (3, 3), (6, 6), (9, 9), (12, 12) $\in$ R [So, reflexive]

(3, 6), (6, 12), (3, 12) $\in$ R [So, transitive].

$\because$ (3, 6) $\in R$ but (6, 3) $\notin R$ [So, non-symmetric]

21. **(d)** Given $f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1}x$

for $x \in (-1, 1)$

$$\text{If } x \in (-1, 1) \Rightarrow \tan^{-1} x \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$$

$$\Rightarrow 2\tan^{-1} x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

Clearly, range of $f(x) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

For f to be onto, codomain = range

$\therefore$ Co-domain of function $= B = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

22. **(c)** $\because$ (1, 1) $\notin R \Rightarrow R$ is not reflexive

$\because$ (2, 3) $\in R$ but (3, 2) $\notin R$

$\therefore$ R is not symmetric

$\because$ (4, 2) and (2, 4) $\in R$ but (4, 4) $\notin R$

$\Rightarrow R$ is not transitive

23. **(a)** Given that $f(x)$ is onto

$\therefore$ range of $f(x)$ = codomain = S

Now, $f(x)=\sin x-\sqrt{3}\cos x+1$

$$=2\sin\left(x-\frac{\pi}{3}\right)+1$$

we know that $-1\le\sin\left(x-\frac{\pi}{3}\right)\le 1$

$$-1\le 2\sin\left(x-\frac{\pi}{3}\right)+1\le 3 \qquad \therefore f(x)\in[-1,\ 3]=S$$

24. **(d)** We have $f:N\to I$

Let x and y are two even natural numbers,

and $f(x)=f(y)\Rightarrow\frac{-x}{2}=\frac{-y}{2}\Rightarrow x=y$

$\therefore f(n)$ is one-one for even natural number.

Let x and y are two odd natural numbers and

$$f(x)=f(y)\Rightarrow\frac{x-1}{2}=\frac{y-1}{2}\Rightarrow x=y$$

$\therefore f(n)$ is one-one for odd natural number.

Hence f is one-one.

Let $y=\frac{n-1}{2}\Rightarrow 2y+1=n$

This shows that n is always odd number for $y\in$ I. ....(i)

and $y=\frac{-n}{2}\Rightarrow -2y=n$

This shows that n is always even number for $y\in$ I. ....(ii)

From (i) and (ii)

Range of f = I = codomain

$\therefore f$ is onto.

Hence f is one one and onto both.

25. **(a)** $y=\frac{8^{2x}-8^{-2x}}{8^{2x}+8^{-2x}}$

$$\frac{1+y}{1-y}=\frac{8^{2x}}{8^{-2x}}\Rightarrow 8^{4x}=\frac{1+y}{1-y}$$

$$\Rightarrow 4x=\log_8\left(\frac{1+y}{1-y}\right)$$

$$\Rightarrow x=\frac{1}{4}\log_8\left(\frac{1+y}{1-y}\right)$$

$$\therefore f^{-1}(x)=\frac{1}{4}\log_8\left(\frac{1+x}{1-x}\right)$$

26. **(b)** $(gof)(x)=g(f(x))=f^2(x)+f(x)-1$

$$g\left(f\left(\frac{5}{4}\right)\right)=4\left(\frac{5}{4}\right)^2-10.\frac{5}{4}+5=-\frac{5}{4}$$

$[\because g(f(x))=4x^2-10x+5]$

$$g\left(f\left(\frac{5}{4}\right)\right)=f^2\left(\frac{5}{4}\right)+f\left(\frac{5}{4}\right)-1$$

$$-\frac{5}{4}=f^2\left(\frac{5}{4}\right)+f\left(\frac{5}{4}\right)-1$$

$$f^2\left(\frac{5}{4}\right)+f\left(\frac{5}{4}\right)+\frac{1}{4}=0$$

$$\left(f\left(\frac{5}{4}\right)+\frac{1}{2}\right)^2=0$$

$$t\left(\frac{5}{4}\right)=-\frac{1}{2}$$

27. **(d)** $f(f(x))=\dfrac{a-\left(\frac{a-x}{a+x}\right)}{a+\left(\frac{a-x}{a+x}\right)}=x$

$$\Rightarrow\frac{a-ax}{1+x}=f(x)\Rightarrow\frac{a(1-x)}{1+x}=\frac{a-x}{a+x}\Rightarrow a=1$$

$$\therefore f(x)=\frac{1-x}{1+x}\Rightarrow f\left(-\frac{1}{2}\right)=3$$

28. **(b)** $\because \phi(x)=((hof)og)(x)$

$$\because\ \phi\left(\frac{\pi}{3}\right)=h\left(f\left(g\left(\frac{\pi}{3}\right)\right)\right)=h(f(\sqrt{3}))=h(3^{1/4})$$

$$=\frac{1-\sqrt{3}}{1+\sqrt{3}}=-\frac{1}{2}(1+3-2\sqrt{3})=\sqrt{3}-2=-(-\sqrt{3}+2)$$

$$=-\tan 15°=\tan(180°-15°)=\tan\left(\pi-\frac{\pi}{12}\right)=\tan\frac{11\pi}{12}$$

29. **(c)** $f(x)=x^2;x\in\mathrm{R}$

$g(\mathrm{A})=\{x\in\mathrm{R}:f(x)\in\mathrm{A}\}\ \mathrm{S}=[0,4]$

$g(\mathrm{S})=\{x\in\mathrm{R}:f(x)\in\mathrm{S}\}$

$=\{x\in\mathrm{R}:0\le x^2\le 4\}=\{x\in\mathrm{R}:-2\le x\le 2\}$

$\therefore\ g(\mathrm{S})\ne\mathrm{S}\ \therefore\ f(g(\mathrm{S}))\ne f(\mathrm{S})$

$g(f(\mathrm{S}))=\{x\in\mathrm{R}:f(x)\in f(\mathrm{S})\}$

$=\{x\in\mathrm{R}:x^2\in\mathrm{S}^2\}=\{x\in\mathrm{R}:0\le x^2\le 16\}$

$=\{x\in\mathrm{R}:-4\le x\le 4\}$

$\therefore\ g(f(\mathrm{S}))\ne g(\mathrm{S})$

$\therefore g(f(\mathrm{S}))=g(\mathrm{S})$ is incorrect.

30. **(a)** The given relation is

$$(f_2 o J o f_1)(x) = f_3(x) = \frac{1}{1-x}$$

$$\Rightarrow (f_2 o J)(f_1(x)) = \frac{1}{1-x}$$

$$\Rightarrow (f_2 o J)\left(\frac{1}{x}\right) = \frac{1}{1-\frac{1}{\frac{1}{x}}} = \frac{\frac{1}{x}}{\frac{1}{x}-1} \quad \left[\because f_1(x) = \frac{1}{x}\right]$$

$$\Rightarrow (f_2 o J)(x) = \frac{x}{x-1} \quad \left[\frac{1}{x} \text{ is replaced by } x\right]$$

$$\Rightarrow f_2(J(x)) = \frac{x}{x-1}$$

$$\Rightarrow 1 - J(x) = \frac{x}{x-1} \quad [\because f_2(x) = 1-x]$$

$$\therefore J(x) = 1 - \frac{x}{x-1} = \frac{1}{1-x} = f_3(x)$$

31. **(c)** Here,

$R_1 = \{(x, y) \in N \times N : 2x + y = 10\}$ and

$R_2 = \{(x, y) \in N \times N : x + 2y = 10\}$

For R_1; $2x + y = 10$ and $x, y \in N$

So, possible values for x and y are:

$x = 1, y = 8$ i.e. $(1, 8)$;

$x = 2, y = 6$ i.e. $(2, 6)$;

$x = 3, y = 4$ i.e. $(3, 4)$ and

$x = 4, y = 2$ i.e. $(4, 2)$.

$R_1 = \{(1, 8), (2, 6), (3, 4), (4, 2)\}$

Therefore, Range of R_1 is $\{2, 4, 6, 8\}$

R_1 is not symmetric

Also, R_1 is not transitive because $(3, 4), (4, 2) \in R_1$ but $(3, 2) \notin R_1$

Thus, options A, B and D are incorrect.

For R_2; $x + 2y = 10$ and $x, y \in N$

So, possible values for x and y are:

$x = 8, y = 1$ i.e. $(8, 1)$;

$x = 6, y = 2$ i.e. $(6, 2)$;

$x = 4, y = 3$ i.e. $(4, 3)$ and

$x = 2, y = 4$ i.e. $(2, 4)$

$R_2 = \{(8, 1), (6, 2), (4, 3), (2, 4)\}$

Therefore, Range of R_2 is $\{1, 2, 3, 4\}$

R_2 is not symmetric and transitive.

32. **(a)** Both R_1 and R_2 are symmetric as

For any $(x, y) \in R_1$, we have

$(y, x) \in R_1$ and similarly for R_2

Now, for R_2, $(b, a) \in R_2$, $(a, c) \in R_2$ but $(b, c) \notin R_2$.

Similarly, for R_1, $(b, c) \in R_1$, $(c, a) \in R_1$ but $(b, a) \notin R_1$.

Therefore, neither R_1 nor R_2 is transitive.

33. **(d)** Suppose $y = f(x)$

$$\Rightarrow \quad y = \frac{x-1}{x-2}$$

$$\Rightarrow \quad yx - 2y = x - 1$$

$$\Rightarrow \quad (y-1)x = 2y - 1$$

$$\Rightarrow \quad x = f^{-1}(y) = \frac{2y-1}{y-1}$$

As the function is invertible on the given domain and its inverse can be obtained as above.

34. **(d)** $f(g(x)) = x$

$$\Rightarrow \quad f(3^{10}x - 1) = 2^{10}(3^{10}.x - 1) + 1 = x$$

$$\Rightarrow \quad 2^{10}(3^{10}x - 1) + 1 = x$$

$$\Rightarrow \quad x(6^{10} - 1) = 2^{10} - 1$$

$$\Rightarrow \quad x = \frac{2^{10}-1}{6^{10}-1} = \frac{1-2^{-10}}{3^{10}-2^{-10}}$$

35. **(c)** $f_1(x) = f_{0+1}(x) = f_0(f_0(x)) = \dfrac{1}{1-\dfrac{1}{1-x}} = \dfrac{x-1}{x}$

$$f_2(x) = f_{1+1}(x) = f_0(f_1(x)) = \frac{1}{1-\frac{x-1}{x}} = x$$

$$f_3(x) = f_{2+1}(x) = f_0(f_2(x)) = f_0(x) = \frac{1}{1-x}$$

$$f_4(x) = f_{3+1}(x) = f_0(f_3(x)) = \frac{x-1}{x}$$

$$\therefore \quad f_0 = f_3 = f_6 = \dots\dots\dots = \frac{1}{1-x}$$

$$f_1 = f_4 = f_7 = \dots\dots\dots = \frac{x-1}{x}$$

$$f_2 = f_5 = f_8 = \dots\dots\dots = x$$

$$f_{100}(3) = \frac{3-1}{3} = \frac{2}{3} \; f_1\left(\frac{2}{3}\right) = \frac{\frac{2}{3}-1}{\frac{2}{3}} = -\frac{1}{2}$$

$$f_2\left(\frac{3}{2}\right) = \frac{3}{2}$$

$$\therefore \quad f_{100}(3) + f_1\left(\frac{2}{3}\right) + f_2\left(\frac{3}{2}\right) = \frac{5}{3}$$

36. **(b)** Since $f(x)$ and $g(x)$ are inverse of each other

$\therefore \quad g'(f(x)) = \dfrac{1}{f'(x)}$

$\Rightarrow \quad g'(f(x)) = 1 + x^5 \qquad \left(\because \ f'(x) = \dfrac{1}{1+x^5}\right)$

Here $x = g(y)$

$\therefore \quad g'(y) = 1 + [g(y)]^5$

$\Rightarrow g'(x) = 1 + (g(x))^5$

37. **(d)** Let A and B be non-empty sets in R.

Let $f : A \to B$ is bijective function.

Clearly statement - 1 is true which says that f is an onto function.

Statement - 2 is also true statement but it is not the correct explanation for statement-1

38. **(a)** Given f is a bijective function

$\therefore \quad f : [1, \infty) \to [1, \infty)$

$f(x) = (x-1)^2 + 1, \ x \ge 1$

Let $\quad y = (x-1)^2 + 1 \Rightarrow (x-1)^2 = y - 1$

$\Rightarrow \ x = 1 \pm \sqrt{y-1} \Rightarrow f^{-1}(y) = 1 \pm \sqrt{y-1}$

$\Rightarrow \quad f^{-1}(x) = 1 + \sqrt{x-1} \ \{\therefore x \ge 1\}$

Hence statement-2 is correct

Now $f(x) = f^{-1}(x)$

$\Rightarrow f(x) = x \Rightarrow (x-1)^2 + 1 = x$

$\Rightarrow \quad x^2 - 3x + 2 = 0 \Rightarrow x = 1, 2$

Hence statement-1 is correct

39. **(d)** Given that $f(x) = (x+1)^2 - 1, \ x \ge -1$

Clearly $D_f = [-1, \infty)$ but co-domain is not given. Therefore $f(x)$ is onto.

Let $f(x_1) = f(x_2)$

$\Rightarrow \quad (x_1 + 1)^2 - 1 = (x_2 + 1)^2 - 1$

$\Rightarrow \quad x_1 = x_2$

$\therefore \quad f(x)$ is one-one, hence $f(x)$ is bijection

$\because \quad (x+1)$ being something +ve, $\forall \, x > -1$

$\therefore \quad f^{-1}(x)$ will exist.

Let $(x+1)^2 - 1 = y$

$\Rightarrow x + 1 = \sqrt{y+1} \qquad$ (+ve square root as $x + 1 \ge 0$)

$\Rightarrow x = -1 + \sqrt{y+1}$

$\Rightarrow f^{-1}(x) = \sqrt{x+1} - 1$

Then $\quad f(x) = f^{-1}(x)$

$\Rightarrow (x+1)^2 - 1 = \sqrt{x+1} - 1$

$\Rightarrow (x+1)^2 = \sqrt{x+1} \ \Rightarrow \ (x+1)^4 = (x+1)$

$\Rightarrow (x+1)[(x+1)^3 - 1] = 0 \ \Rightarrow \ x = -1, 0$

$\therefore$ The statement-1 and statement-2 both are true.

40. **(d)** Clearly $f(x) = 4x + 3$ is one one and onto, so it is invertible.

Let $\quad f(x) = 4x + 3 = y$

$\Rightarrow x = \dfrac{y-3}{4} \qquad \therefore \ g(y) = \dfrac{y-3}{4}$